Chapter 4/Straightening Curves and Plots

The Idea of Straightening out Curves

When we deal with closely related variables, some advantages occur if we can express their relationship linearly. Interpolation and interpretation are relatively easier, and departures from the fit are more clearly detected. In this chapter, we offer ways of straightening out curves.

When we have empirically determined relations between variables, we cannot hope that straightening the relation within the range of the observed data will also straighten it far beyond the observed range. Although luck might help us, ordinarily we need some sort of theory or previous experience to do that.

This chapter helps us re-express one or both of a pair of variables so that relations originally curved are straighter. If y and x are the variables, we consider re-expressing y or x or both. The primary tools are:

1. a ladder of re-expressions, and
2. rules for determining which direction to move on the ladder.

Learning the techniques will be simplified if at first we concentrate on straightening out a functional relation, and then later extend the idea to scatter plots and other empirical data.

4A. The Ladder of Re-expressions

Since we need a systematic set of re-expressions, the powers of a variable naturally suggest themselves. As a start, let the power p take the values

$$-3, \quad -2, \quad -1, \quad -\tfrac{1}{2}, \quad \#, \quad \tfrac{1}{2}, \quad 1, \quad 2, \quad 3.$$

(More about # later.) Let us think about only positive values of the variable, which for convenience we call t.

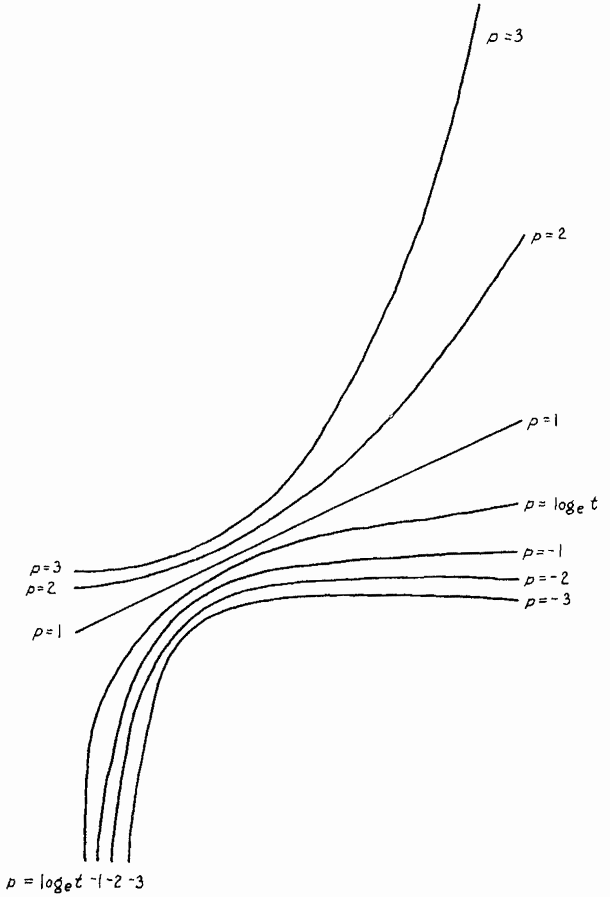
First, we want a set of re-expressions each of which is monotonic in the same direction. When $p > 0$, as t increases, t^p increases. When $p < 0$, as t increases, t^p decreases. To make them all increase as t increases, we can use $-t^p$ when p is negative.

Second, what shapes do these curves have? When $p > 1$, they are hollow upward $\smile$. When $p = 1$, the curve is straight. When $p < 1$, the curves are hollow downward ⌒. Exhibit **1** shows the shapes of these curves, although they have been rescaled as in Exhibit 8 of Chapter 5 and pulled apart by additive constants, so that they can be seen more clearly.

What do we choose for #? The value $p = 0$ leads to a constant, and so we cannot usefully choose it. Instead we choose $\log t$. (We might think of these powers of t as coming from $\int t^{p-1}\, dt$. When $p = 0$, we get $\log t$.) Some may want to do something else, and they might get a different answer. The $\log t$ curve fits in well and we wouldn't want to leave out the logarithm because it is the

Exhibit **1** of Chapter 4

Shapes of curves $z = t^p$ for $p = -3, -2, \#, 1, 2, 3$.



transformation most commonly used. Consequently, at # we do not use t^0, but log t.

When we are trying re-expressions, we will move up and down the ladder of re-expressions given here, searching for one that straightens well. To aid our intuition about which direction to move on the ladder, we will do one example where we have complete command of the information. Then we will give a set of rules.

4B. Re-expressing $y = x^2$

As our instructive example, we choose the functional relation

$$y = x^2, \qquad x \geq 0.$$

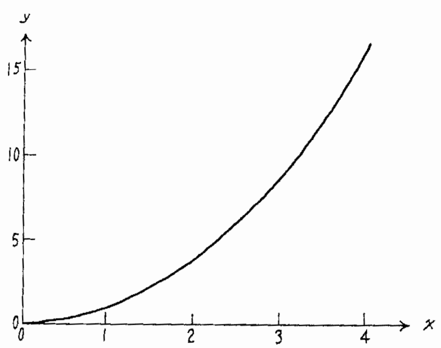
Its graph is shown in Exhibit **2,** and we note that it is hollow upward and increasing as x increases.

1. *What re-expression of y would straighten out the curve?* Let us consider what would happen if we replaced y by either y^2 or $y^{1/2}$. We would be helped by considering points in two intervals 0 to 1 and 1 to 4, because all t^p are equal at $t = 1$.

Using y^2. If y is replaced by y^2, then all the points for $0 < x < 1$ will be lower than they were before, because squaring a number between 0 and 1 makes it smaller. Squaring the y's where $x > 1$ makes the y's larger. We have, all told, then, bent the curve more than before.

Exhibit **2** of Chapter 4

Graph of $y = x^2$ hollow upwards, y increasing as x increases.



Three points, two slopes. Let's make this more quantitative by a rough device. We could compute the slope of the chord from the origin to the value at $x = 1$ and the slope from the point at $x = 1$ to $x = 4$ for both the original curve and the transformed one. Ideally the slopes are equal for the two intervals.

Original slopes:	$0 < x < 1,$	$1/1 = 1;$	$1 < x < 4,$	$15/3 = 5;$
New slopes:	$0 < x < 1,$	$1/1 = 1;$	$1 < x < 4,$	$255/3 = 85.$

The original slopes are 1 and 5, a ratio of 5, and the new slopes are 1 and 85, a ratio of 85. Using y^2 has not moved the slopes closer together but farther apart. We are moving in the wrong direction.

Next let us try $y^{1/2} = \sqrt{y}$. Now, points in the interval $0 < x < 1$ are **increased,** because the square root—indeed, *any* root with power between 0 and 1—*raises* such values; for example,

$$\sqrt{0.01} = 0.1, \qquad \sqrt[3]{0.001} = 0.1.$$

In the interval $1 < x < 4$, the numbers become **smaller.** Thus, we are raising the lefthand set of numbers and decreasing the righthand ones, a possible step in the correct direction. Thus, if we replace y by $y^* = \sqrt{y}$ we get the relation

$$y^* = \sqrt{x^2}$$

or

$$y^* = x, \qquad x > 0,$$

and this is an equation of a **straight line** through the origin.

The suggestion we want to draw from this example is that for hollow-upward, monotonically increasing curves, if we want to replace y, we should move down the ladder to a p smaller than 1. We do not draw the lesson that this will work well or that we know how far to go. In the present example, the idea of either squaring or square-rooting sticks out because we knew the formula.

Let us now try for a second lesson by going back to the original curve $y = x^2$.

2. *What re-expression of x would straighten out the curve?* Again, our knowledge of the functional form suggests that we replace x by either $x^* = x^2$ or $x^* = \sqrt{x}$. If we replace x by $x^* = x^2$, then we will have points (x^2, y), and since $y = x^2$, we again get a straight line because the points are (x^2, x^2). We have moved up the ladder. We are tacitly assuming that all powers p are available, but relatively few are used. Let us suppose that only the powers -3, -2, -1, #, 1, 2, 3 had been available. What would we try for y?

3. *Trying* log *y*. Since we want to go down the ladder for y, let us replace y by $\log y$ and see what happens. We use $\log_e$. We have $\log 0 = -\infty$, $\log 1 = 0$, $\log 4 = 1.39$.

Original slopes:	$0 < x < 1,$	$1;$	$1 < x < 4,$	$5;$
New slopes:	$0 < x < 1,$	$\infty;$	$1 < x < 4,$	$0.46.$

The logarithm increased the slope for the lefthand interval and decreased it for the righthand one, both moves in the correct direction, but it overcorrected.

Starting. We could have avoided the infinities here if we had added a constant to y before we started. Let's ask what constant we could have added to make the slopes of the two chords equal. We want c so that

$$\frac{\log(1+c) - \log c}{1} = \frac{\log(16+c) - \log(1+c)}{3}$$

$$\log\frac{1+c}{c} = \tfrac{1}{3}\log\left(\frac{16+c}{1+c}\right).$$

Trying a few values suggests that $c = 0.95$ gives a close approximation. Ordinarily, we would round this c off to 1, but let us go ahead with 0.95. We are ready to replace

$$y \qquad \text{by} \qquad y^* = \log(y + 0.95).$$

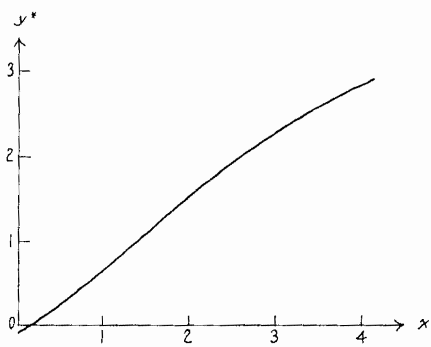
Tabular values to two decimals for $x = 0, 1, 2, 3, 4$ are

x	0	1	2	3	4
y^*	−0.05	0.67	1.60	2.30	2.83

The graph is shown in Exhibit **3.** On the one hand, the curve is not straight, but on the other hand it is much straighter than it was to begin with, and it might

Exhibit **3** of chapter 4

Graph of $y^* = \log(x^2 + 0.95)$.



serve us very well. A straight line could be fitted to this curve that would not be off more than 0.1 in the vertical direction over the range $0 < x < 4$.

The suggestion here is that perfection may not be necessary. A coarse grid on p might be quite satisfactory. Usually we include $p = \frac{1}{2}$ and $p = -\frac{1}{2}$; sometimes $p = \frac{1}{3}$, and even other fractional powers. How much detail is worthwhile depends upon the example being treated.

Although we have not proved it, the direction of the hollowness—upward or downward—and the direction of the monotonicity lead us to a set of rules for making the fit. We give these rules without further discussion in the next section.

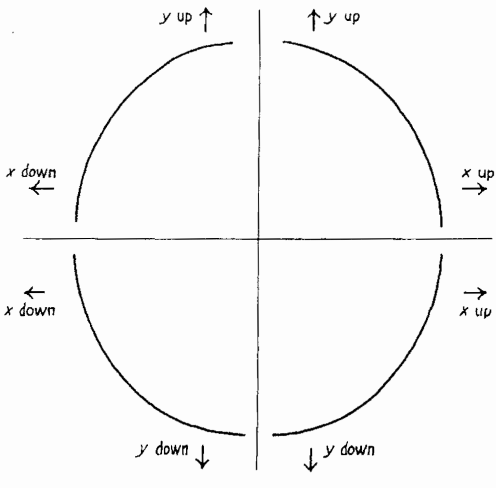
4C. The Bulging Rule

The fundamental rule is to move on the ladder in the direction in which the bulge points. The bulge points separately for x and for y. Exhibit **4** is a reminder of how to use the ladder of powers to aid re-expression. The arcs illustrate four combinations of slope and curvature, four kinds of bulging.

Let us try another example. Data for it are given in Exhibit **5,** and these data are plotted in Exhibit **6.**

Exhibit **4** of Chapter 4

Directions to move. The arrows point in the direction of the bulge for each type of curve and for each variable separately.



Example 1. Use the ladder to straighten out the curve in Exhibit 6 by re-expressing y.

Solution. From the point of view of y, the curve bulges upward, and we want to go up the ladder from $p = 1$.

Let us pick out three points and compute the pairs of slopes. Let us choose $x = 0.1, 3, 9$. The original slopes of the chords are

$$0.1 < x < 3, \quad \frac{2.88 - 0.93}{3 - 0.1} = 0.67; \qquad 3 < x < 9, \quad \frac{4.16 - 2.88}{9 - 3} = 0.21;$$

$$\text{Ratio of slopes:} \quad 0.67/0.21 = 3.2$$

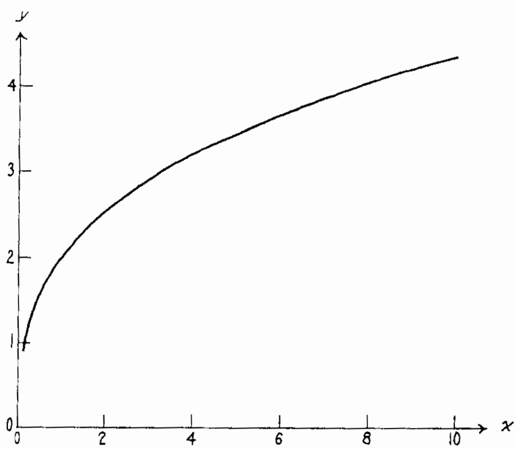
Exhibit **5** of Chapter 4

Data relating *y* and *x* for Example 1, Section 4C

x	y	x	y
.1	.93	1	2.00
.3	1.34	3	2.88
.5	1.59	5	3.42
.7	1.78	7	3.83
.9	1.93	9	4.16

Exhibit **6** of Chapter 4

Plot of data for Example 1.



Let us now move up the y ladder to y^2.

x	0.1	3	9
$y^* = y^2$	0.86	8.29	17.31

The new slopes are

$$0.1 < x < 3, \quad \frac{8.29 - 0.86}{3 - 0.1} = 2.56; \qquad 3 < x < 9, \quad \frac{17.31 - 8.29}{9 - 3} = 1.50;$$

$$\text{Ratio of slopes:} \quad 2.56/1.50 = 1.7.$$

The move has reduced the ratio, but we have not gone far enough. Let us try y^3.

x	0.1	3	9
$y^{**} = y^3$	0.80	23.89	72.0

$$0.1 < x < 3, \quad \frac{23.89 - 0.80}{3 - 0.1} = 7.97; \qquad 3 < x < 9, \quad \frac{72.0 - 23.9}{9 - 3} = 8.01;$$

$$\text{Ratio of slopes:} \quad 7.97/8.01 = 0.995.$$

This is within rounding error of 1.00, and we conclude that replacing y by y^3 will straighten out the curve.

Since we secretly know that the original data were rounded values of $y = 2\sqrt[3]{x}$, we can see that the movement on the ladder has actually found exactly the right re-expression. Nevertheless, it will be instructive to see what would have happened if we had tried instead to re-express x.

Example 2. Straighten out the curve of Exhibit 6 by re-expressing x.

Solution. The curve bulges left, and so we want to go down the ladder to $\log x$ or $-1/x$. Let us try these values, using the same 3 choices of x:

y	0.93	2.88	4.16
x	0.1	3	9
$x^* = \log x$	−1	0.48	0.95
$x^{**} = -1/x$	−10	−0.33	−0.11

As the calculations below show, the ratio of the slopes increases steadily.

	Left interval	Right interval	Ratio	log ratio
x	$\frac{2.88 - 0.93}{3 - 0.1} = 0.67$	$\frac{4.16 - 2.88}{9 - 3} = 0.21$	0.31	−0.51
x^*	$\frac{2.88 - 0.93}{0.48 - (-1)} = 1.32$	$\frac{4.16 - 2.88}{0.95 - 0.48} = 2.72$	2.06	0.31
x^{**}	$\frac{2.88 - 0.93}{-0.33 - (-10)} = 0.20$	$\frac{4.16 - 2.88}{-0.11 - (-0.33)} = 5.82$	29.1	1.46

Moving to the logarithm was already too far.

If we plot the logs of the ratios against p for the three points, we can interpolate to get an estimate of the re-expression that straightens most. The estimate is 0.38, or about $\frac{1}{3}$ (the exact result).

4D. More Complicated Curves

If a curve had a lazy-S shape, we would not be likely to remove the curvature by the process we have described. We might conceivably break it up into two pieces at the inflection point  and fit the two pieces separately. More complicated curves could be treated similarly. Sometimes we can do better than this.

Sometimes we may have theory to guide us, and then we would expect to use it. For example, if we knew that the number of primes among the integers less than x was $x/\log_e x$, approximately, then plotting the observed number y against $x/\log_e x$ would be likely to produce the desired linearity.

4E. Scatter Plots

When the data are not as smooth as those we have dealt with, we replace values in a narrow array by some average—the median or the mean of both y and of x, and then we work with three of these points at a time, as before. The median has some advantages here. The median of the re-expressed values is the re-expressed value of the original median, whereas the mean of the re-expressed values is not the re-expressed mean. Ordinarily it is the mean that possesses such attractive commutative properties, but nonlinearity is not one of those situations.

Summary: Straightening Curves

Straightening out the relation of y to x over the range where we have data need not ensure straightening out the relation outside that range.

"Straightening" is to be attacked initially in terms of three well-selected points.

We use the ratio of the slopes of the two segments connecting the middle (according to x) point to the upper and lower points, planning to bring this ratio close to 1.0 by trying various rungs on the ladder of re-expression, and, where helpful, interpolating.

If a point-cloud (scatter plot) appears otherwise too fuzzy for effective straightening, we divide the (x, y) points into groups according to their x-values, then find (x-median, y-median) for each group, and work with these latter points.

The simplest re-expressions for amounts form a ladder, mainly consisting of t^p for various p, but with $\log t$ taking the place that would otherwise be reserved for t^0 (which is everywhere $= 1$ and thus is unhelpful).

When our curve or point-cloud is still curved, we move on the ladder of re-expression in the direction that the bulge points, a rule that can be applied, with different consequences, to re-expressing either x or y.